$= V(y) - V(x)$

$|f(y) - f(x)| \leq V(y) - V(x)$

$\therefore$ V is continuous at x, as y tends to x, [by equation (5)]

we can find $V(y) - V(x) < \epsilon$

letting $y \to x+$

$\Rightarrow |f(x+) - f(x)| < V(x+) - V(x)$

$\Rightarrow |f(x+) - f(x)| < \epsilon$

$\therefore$ Right hand limit of 'f' at x exist and

$f(x+) = f(x)$

llly, we can prove that

$0 \leq |f(x) - f(x-)| < \epsilon$

$\therefore$ Left hand limit of 'f' at x exist and

$f(x-) = f(x)$

$\therefore$ f is continuous at $x \in (a,b)$

Thus the theorem is true for all interior points of (a,b)

Trivial modifications are needed to prove the theorem at end points.

Theorem 16:

Let f be continuous on [a,b]. then 'f' is of bounded variation on [a,b] iff 'f' can be expressed as the difference of two increasing continuous functions.

proof: Assume that 'f' is continuous on [a,b] and 'f' is of bounded variation on [a,b]

To prove that: 'f' can be expressed as the difference of two increasing continuous functions.

Let V be a function defined as follows

$$V(x) = V_f(a,x), \text{ if } a < x \leq b$$

and $V(a) = 0$

then V and $V-f$ are increasing functions [By thm 12]

Since f is continuous on $[a,b]$

V is also continuous on $[a,b]$ [By thm 14]

Since difference of two continuous functions is also continuous, $V-f$ is also continuous.

Thus, $f = V-(V-f)$

(i.e) We have expressed f as the difference of two continuous increasing functions.

Conversely,

Suppose that f can be expressed as difference of two increasing continuous functions.

(i.e) Let $f = f_1 - f_2$ where f_1 and f_2 are continuous on $[a,b]$

TP: f is continuous on $[a,b]$ and

f is of bounded variation on $[a,b]$

Since f_1 and f_2 are increasing on $[a,b]$

f_1 and f_2 are of bounded variation on $[a,b]$ (By thm 7)

Then $f_1 - f_2 = f$ is of bounded variation on $[a,b]$ [By thm 9]

Since f_1 and f_2 are continuous, their difference $f_1 - f_2 = f$ is also continuous.

Thus f is continuous on $[a,b]$ and f is of bdd variation on $[a,b]$

Note: A continuous functions need not be of bounded variation

Define $f(x) = \begin{cases} x\sin \pi/x, & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$

$P_n = \{0, \frac{2}{2n+1}, \frac{2}{2n-1}, \dots \frac{2}{5}, \frac{2}{3}, 1\}$

Riemann Stieltjes Integral:

Partition:

A partition P for $[a,b]$ is a finite set of points say $P = \{x_0, x_1, x_2, \ldots x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Refinement of a partition:

A partition P' is called the refinement of P (or) if P' is finer than P if $P' \supseteq P$

Norm of a partition:

Norm of a partition P of $[a,b]$ is the length of the largest subinterval of $[a,b]$. It is denoted by $\|P\|$.

Note:

(i) If P' is finer than P then $\|P'\| \leq \|P\|$

(ii) We will be considering the real valued functions f, g, α, β defined and bounded on $[a,b]$ and $[a,b]$ is compact.

(iii) Let α be a function defined on $[a,b]$

The Symbol $\Delta\alpha_k$ is defined as

$$\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$$

then $$\sum_{k=1}^{n} \Delta\alpha_k = \alpha(x_n) - \alpha(x_0)$$

$$= \alpha(b) - \alpha(a)$$

iv) Set of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$

Riemann Stieltjes Sum:

Let $P = \{x_0, x_1, \ldots x_n\}$ be a partition of $[a,b]$. Let t_k be a point in the subinterval $[x_{k-1}, x_k]$

Then the Sum of the form.

(2)

$$\boxed{S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\, d\alpha_k}$$ is called Riemann stieltjes

Sum of 'f' w.r.t 'α'

Riemann Stieltjes integrable:

We Say 'f' is Riemann integrable w.r.t α on $[a, b]$ and we write $f \in R(\alpha)$, $[a, b]$ if there exists a number 'A' satisfies the following property.

For any given $\epsilon > 0$ $\exists$ a partition on $[a, b]$ such that for every partition P finer than P_ϵ

i.e) $P_\epsilon \subseteq P$ and for every choice of t_k in $[x_{k-1}, x_k]$

we have,

$$|S(P, f, \alpha) - A| < \epsilon \Rightarrow f \in R(\alpha) \text{ and}$$

$$A = \int_a^b f(x)\, d\alpha(x)$$

We say that when such a number exists it is uniquely determined and is denoted by, $\int_a^b f\, d\alpha$ (or) $\int_a^b f\, d\alpha(x)$

And we also say that Riemann stieltjes integral $\int_a^b f\, d\alpha$ exists

The functions f and α are refferred to as integrated and integrator respectively.

Note:

In Special case, when $\alpha(x) = x$, we write $S(P, f)$ instead of $S(P, f, \alpha)$ and $f \in R$ instead of $f \in R(\alpha)$.

The integral is then called a Riemann itegral and is denoted by $\int_a^b f\, dx$ (or) $\int_a^b f(x)\, dx$.

Linear Properties:

Theorem: 1 Change of integral

If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a,b]$ for any two constants c_1, c_2.

And we have

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha$$

Proof:

Given that $f \in R(\alpha)$ on $[a,b]$ and $g \in R(\alpha)$ on $[a,b]$

T.P: $c_1 f + c_2 g \in R(\alpha)$ on $[a,b]$

Let $c_1 f + c_2 g = h$

Let P be the partition on $[a,b]$

Now consider

$$S(P,h,\alpha) = \sum_{k=1}^{n} h(t_k)\, \Delta\alpha_k$$

$$= \sum_{k=1}^{n} ((c_1 f + c_2 g)(t_k))\, \Delta\alpha_k$$

$$= \sum_{k=1}^{n} c_1 f(t_k)\Delta\alpha_k + \sum_{k=1}^{n} c_2 g(t_k)\, \Delta\alpha_k$$

$$= c_1 \sum_{k=1}^{n} f(t_k)\Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\, \Delta\alpha_k$$

$$= c_1 S(P,f,\alpha) + c_2 S(P,g,\alpha) \longrightarrow (1)$$

Since $f \in R(\alpha)$ on $[a,b]$

For any given $\epsilon > 0$ $\exists$ a partition P'_ϵ on $[a,b]$ such that $\forall$ $P \supseteq P'_\epsilon$ and $t_k \in (x_{k-1}, x_k)$ we have

$$\left| S(P,f,\alpha) - \int_a^b f\, d\alpha \right| < \epsilon \longrightarrow (2)$$

$|||^{ly}$ Since $g \in R(\alpha)$ on $[a,b]$

For any given $\epsilon > 0$ $\exists$ P''_ϵ on $[a,b]$ such that

$P \geq P_\epsilon''$ and $t_k \in (x_{k-1}, x_k)$

we have

$$\left| S(P, g, \alpha) - \int_a^b g\, d\alpha \right| < \epsilon \longrightarrow (3)$$

Let $$A = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha \longrightarrow (4)$$

$$P_\epsilon = P_\epsilon' \cup P_\epsilon''$$

then we have if P is finer than P_ϵ,

$$P \geq P_\epsilon' \text{ and } P \geq P_\epsilon''$$

for such 'P' equation (2) and (3) hold good.

Then for every partition $P \geq P_\epsilon$.

$$|S(P, h, \alpha) - A| = \left| c_1 S(P, f, \alpha) + c_2 S(P, g, \alpha) - c_1 \int_a^b f\, d\alpha - c_2 \int_a^b g\, d\alpha \right|$$

$$= \left| c_1 \left[S(P, f, \alpha) - \int_a^b f\, d\alpha \right] + c_2 \left[S(P, g, \alpha) - \int_a^b g\, d\alpha \right] \right|$$

$$\leq |c_1| \left| S(P, f, \alpha) - \int_a^b f\, d\alpha \right| + |c_2| \left| S(P, g, \alpha) - \int_a^b g\, d\alpha \right|$$

$$< |c_1| \epsilon + |c_2| \epsilon = (|c_1| + |c_2|) \cdot \epsilon$$

$\therefore h \in R(\alpha)$ on $[a, b]$ and $A = \int_a^b h\, d\alpha$

ie, $c_1 f + c_2 g \in R(\alpha)$ on (a, b)

And $$c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha = \int_a^b (c_1 f + c_2 g)\, d\alpha$$

Theorem (2)

If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$ then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ for any two constants c_1 and c_2 and we have $$\int_a^b f\, d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta$$

proof: Given that $f \in R(\alpha)$ and $f \in R(\beta)$

TP: $f \in R(c_1\alpha + c_2\beta)$

Let $\gamma = c_1\alpha + c_2\beta$

Let P be a partition on $[a,b]$

Now consider

$$S(P,f,\gamma) = \sum_{k=1}^{n} f(t_k)\,\Delta\gamma_k$$

$$= \sum_{k=1}^{n} f(t_k)\{\gamma(x_k)-\gamma(x_{k-1})\}$$

$$= \sum_{k=1}^{n} f(t_k)\{(c_1\alpha+c_2\beta)(x_k)-(c_1\alpha+c_2\beta)(x_{k-1})\}$$

$$= \sum_{k=1}^{n} f(t_k)\{c_1[\alpha(x_k)-\alpha(x_{k-1})]$$

$$+ c_2[\beta(x_k)-\beta(x_{k-1})]$$

$$= c_1\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k + c_2\sum_{k=1}^{n} f(t_k)\,\Delta\beta_k$$

$$S(P,f,\gamma) = c_1 S(P,f,\alpha) + c_2 S(P,f,\beta) \longrightarrow (1)$$

Since $f\in R(\alpha)$ on $[a,b]$

For any $\epsilon>0$ $\exists$ a partition P_ϵ' such that $P \supseteq P_\epsilon'$ and $t_k \in (x_{k-1}, x_k)$ we have

$$\left|S(P,f,\alpha) - \int_a^b f\,d\alpha\right| < \epsilon \longrightarrow (2)$$

Let $A = c_1\int_a^b f\,d\alpha + c_2\int_a^b f\,d\beta$

Let $P_\epsilon = P_\epsilon' \cup P_\epsilon''$

then $P \supseteq P_\epsilon \Rightarrow P_\epsilon' \subseteq P$, $P \supseteq P_\epsilon''$

For such 'P' equations (2) and (3) hold good.

Then for every $P \supseteq P_\epsilon$,

$$|S(P,f,\gamma)-A| = \left|S(P,f,\gamma) - c_1\int_a^b f\,d\alpha - c_2\int_a^b f\,d\beta\right|$$

$$= \left|c_1(S(P,f,\alpha) + c_2 S(P,f,\beta) - c_1\int_a^b f\,d\alpha - c_2\int_a^b f\,d\beta\right|$$

$$\leq |c_1| \left| S(P, f, \alpha) - \int_a^b d\alpha \right| + |c_2| \left| S(P, f, \beta) - \int_a^b f\, d\beta \right|$$

$$< |c_1| \left| S(P, f, \alpha) - \int_a^b d \right| \quad ⑥$$

$$< |c_1| \epsilon + |c_2| \epsilon = [|c_1| + |c_2|] \epsilon$$

Since ϵ is of our choice, we can take

$$(|c_1| + |c_2|) \epsilon = \epsilon$$

Then $f \in R(\gamma)$ on (a, b) and $A = \int_a^b f\, d\gamma$

$\Rightarrow f \in R(c_1\alpha + c_2\beta)$ on (a, b) and

$$c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta = \int_a^b f\, d\gamma$$

Theorem ③:

Assume that $c \in (a, b)$ if two of three integrals in (1) exist then the third also exist and we have

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha \longrightarrow (1)$$

Proof:
Assume that $\int_a^c f\, d\alpha$ and $\int_c^b f\, d\alpha$ exist

To prove that: $\int_a^b f\, d\alpha$ exist

i.e.) $f \in R(\alpha)$ on $[a, c]$ and $f \in R(\alpha)$ on $[c, b]$

To prove: $f \in R(\alpha)$ on $[a, b]$

Let P be a partition of $[a, b]$ such that $c \in P$

Let P' and P'' are partitions of $[a, c]$ and $[c, b]$ respectively.

Then the Riemann stieltjes sum of these 3 partition are connected by the relation.

$$S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha) \longrightarrow (2)$$

Since $\int_a^c f\,d\alpha$ exists, $f \in R(\alpha)$ on $[a,c]$

we have for any given $\epsilon > 0$ $\exists$ a partition P'_ϵ so that, $P' \supseteq P'_\epsilon$ we have,

$$\left|S(P', f, \alpha) - \int_a^c f\,d\alpha\right| < \epsilon/2 \longrightarrow ③$$

Since $\int_c^b f\,d\alpha$ exists, $f \in R(\alpha)$ on $[c,b]$

we have for any $\epsilon > 0$ $\exists$ a partition P''_ϵ so that $P'' \supseteq P''_\epsilon$ we have,

$$\left|S(P'', f, \alpha) - \int_c^b f\,d\alpha\right| < \epsilon/2 \longrightarrow ④$$

Let $P_\epsilon = P'_\epsilon \cup P''_\epsilon$

Then P_ϵ is a partition of $[a,b]$

Let P is a partition finer than P_ϵ

i.e.) $P \supseteq P_\epsilon$

$\Rightarrow P \cap [a,c] \supseteq P_\epsilon \cap [a,c]$

$\Rightarrow$ $P' \supseteq P'_\epsilon$

$|||^{ly}$ $P'' \supseteq P''_\epsilon$

For such P, ③ & ④ hold

Let $A = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

For every $P \supseteq P_\epsilon$

$$|S(P, f, \alpha) - A| = \left|S(P', f, \alpha) + S(P'', f, \alpha) - \int_a^c f\,d\alpha - \int_c^b f\,d\alpha\right|$$

$$\leq \left|S(P', f, \alpha) - \int_a^c f\,d\alpha\right| + \left|S(P'', f, \alpha) - \int_c^b f\,d\alpha\right|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$\therefore f \in R(\alpha)$ on $[a,b]$ and $A = \int_a^b f\,d\alpha$

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$$\therefore A = \int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

$$(ie)\ \int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

Similarly the other two cases can also be proved.

Integration by parts

Theorem:

If $f \in R(\alpha)$ on $[a,b]$ then $\alpha \in R(f)$ on $[a,b]$

and we have,

$$\int_a^b f(x)\,d[\alpha(x)] + \int_a^b \alpha(x)\,d[f(x)] = f(b)\,\alpha(b) - f(a)\,\alpha(a)$$

Proof: Given that $f \in R(\alpha)$ on $[a,b]$

To prove that: $\alpha \in R(f)$ on $[a,b]$

Since $f \in R(\alpha)$ on $[a,b]$

for any given $\epsilon > 0$ $\exists$ a partition P_ϵ of $[a,b]$ such that $P' \supseteq P_\epsilon$ we have

$$\left| S(P', f, \alpha) - \int_a^b f\,d\alpha \right| < \epsilon \longrightarrow (1)$$

Let $A' = f(b)\,\alpha(b) - f(a)\,\alpha(a)$

then $A' = \sum_{k=1}^{n} f(x_k)\,\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\,\alpha(x_{k-1}) \longrightarrow (2)$

Let P be a partition finer than P_ϵ

then $S(P, \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)\,\Delta f_k$, $t_k \in [x_{k-1}, x_k]$

$$\Rightarrow S(P, \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)\,[f(x_k) - f(x_{k-1})]$$

$$= \sum_{k=1}^{n} \alpha(t_k)\,f(x_k) - \sum_{k=1}^{n} \alpha(t_k)\,f(x_{k-1}) \longrightarrow (3)$$

from (2) and (3) we have

$$(2) - (3) \Rightarrow A' - S(P, \alpha, f) = \sum_{k=1}^{n} f(x_k)\,\{\alpha(x_k) - \alpha(t_k)\}$$

$$+ \sum_{k=1}^{n} f(x_{k-1})\,\{-\alpha(x_{k-1}) + \alpha(t_k)\}$$